逆強単調写像に関する変分不等式問題を扱ったBadrievとZadvornovの結果の一考察

Note on Badriev and Zadvornov's results for variational inequality problems for inverse-strongly monotone mappings

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Abstract

In [3], Badriev and Zadvornov consider a variational inequality problem with two monotone mappings A and B. The authors show that an iterative sequence converges weakly to a solution of the problem under suitable conditions for A and B. In this paper, to apply a theorem in [6], we show an iterative sequence which converges strongly to a solution of the problem under same conditions for A and B in [3].

Keywords: Fixed point, variational inequality problem, inverse-strongly monotone mapping.

1 Introduction

Let V and H be Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_H$, respectively. Let $A : V \to V$ and $B : H \to H$ be mappings and $F : V \to (-\infty,\infty]$ and $G : H \to (-\infty,\infty]$ be proper continuous lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of Λ, i.e., $\langle\Lambda^* x, v\rangle_V = \langle x, \Lambda v\rangle_H$ for all $x \in H$ and $v \in V$. In [3], Badriev and Zadvornov consider the following variational inequality problem. Find $u \in V$ such that

$$\langle Au, v-u\rangle_V + \langle\Lambda^* B\Lambda u, v-u\rangle_V + G(\Lambda v) - G(\Lambda u) + F(v) - F(u) \geq 0 \quad (1)$$

for all $v \in V$. The authors show that an iterative sequence converges weakly to a solution of the problem under suitable conditions for A and B.

In this paper, to apply a theorem in [6], we show an iterative sequence which converges strongly to a solution of the problem under same conditions for A and B in [3]. The theorem in [6] is related to the result of our previous paper [1].

2 Preliminaries

Let $A : V \to V$ and $B : H \to H$ be mappings. Mappings $A : H \to H$ and $B : V \to V$ are inverse-storongly monotone mappings if there exists $\sigma_A, \sigma_B > 0$ such that

$$\langle Au - Av, u - v\rangle_V \geq \sigma_A \|Au - Av\|_V^2$$

for all $u, v \in V$ and

$$\langle Bx - By, x - y\rangle_H \geq \sigma_B \|Bx - By\|_H^2$$

for all $x, y \in H$. Then A is called σ_A-inverse-strongly monnotone and B is called σ_B-inverse-strongly monotone. Let C be a subset of H. A mapping T of C into itself is called nonexpansive if

$$\|Tx - Ty\|_H \leq \|x - y\|_H$$

for all $x, y \in C$. We denote by $\mathrm{F}(T)$ the set of fixed points of T. Let f be a functional on H. By P_f, we denote the proximal mapping which takes each $x \in H$ to the element $y = P_f(x)$ that is a solution of

$$\langle y - x, z - y \rangle_H + f(z) - f(y) \geq 0$$

for all $z \in H$. P_f satisfies the following.

$$\|P_f x - P_f y\|_H^2 \leq \langle P_f x - P_f y, x - y \rangle_H \quad (2)$$

for all $x, y \in H$.

To introduce our main result, we need the following theorems. Theorem 1 is the result of Badriev and Zadvornov in [3]. For the sake of completeness, we show the proof in Section 4.

Theorem 1. *Let V and H be Hilbert spaces. Let $A : V \to V$ be a σ_A-inverse-storongly monotone mapping and $B : H \to H$ be a σ_B-inverse-strongly monotone mapping. Let $F : V \to (-\infty, \infty]$ and $G : H \to (-\infty, \infty]$ be proper convex lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of Λ, i.e., $\langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H$ for all $x \in H$ and $v \in V$. In addition, we assume that the operator $\Lambda^*\Lambda$ is a canonical isomorphism, i.e., $v = \Lambda^*\Lambda v$ for all $v \in V$. Let $Q = V \times H \times H$ be the Hilbert space with inner product*

$$\langle \cdot, \cdot \rangle_Q = \frac{1 - \tau_A r}{\tau_A} \langle \cdot, \cdot \rangle_V + \frac{1}{\tau_B} \langle \cdot, \cdot \rangle_H + \frac{1}{r} \langle \cdot, \cdot \rangle_H,$$

where τ_A, τ_B and r are positive constants satisfying $\tau_A r < 1$. Let $T : Q \to Q$ be a mapping defined by $Tq = (T_1 q, T_2 q, T_3 q)$, where

$$T_1 q = P_{\tau_A F}(q_1 - \tau_A(Aq_1 + \Lambda^* q_3 + r\Lambda^*(\Lambda q_1 - q_2))),$$

$$T_2 q = P_{\tau_B G}(q_2 - \tau_B(Bq_2 - q_3 + r(q_2 - \Lambda T_1 q))),$$

$$T_3 q = q_3 + r(\Lambda T_1 q - T_2 q)$$

for $q = (q_1, q_2, q_3) \in Q$. Let $q = (u, y, \lambda)$. Then q is a fixed point of T if and only if

$$\begin{cases} -Au - \Lambda^*\lambda \in \partial F(u), \\ \lambda - By \in \partial G(y), \\ y = \Lambda u. \end{cases}$$

Moreover, u is a solution of the problem (1).

Theorem 2 is the result of Badriev and Zadvornov in [3]. For the sake of completeness, we show the proof in Section 4.

Theorem 2. *Let V and H be Hilbert spaces. Let $A : V \to V$ be a σ_A-inverse-storongly monotone mapping and $B : H \to H$ be a σ_B-inverse-strongly monotone mapping. Let $F : V \to (-\infty, \infty]$ and $G : H \to (-\infty, \infty]$ be proper convex lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of Λ, i.e., $\langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H$ for all $x \in H$ and $v \in V$. In addition, we assume that the operator $\Lambda^*\Lambda$ is a canonical isomorphism, i.e., $v = \Lambda^*\Lambda v$ for all $v \in V$. Let $Q = V \times H \times H$ be the Hilbert space with inner product $\langle \cdot, \cdot \rangle_Q = \frac{1-\tau_A r}{\tau_A} \langle \cdot, \cdot \rangle_V + \frac{1}{\tau_B} \langle \cdot, \cdot \rangle_H + \frac{1}{r} \langle \cdot, \cdot \rangle_H$, where τ_A, τ_B and r are positive constants satisfying $\tau_A r < 1$,*

$$\tau_A < \frac{2\sigma_A}{2\sigma_A r + 1} \quad \text{and} \quad \tau_B < \frac{2\sigma_B}{2\sigma_B r + 1}.$$

Let $T : Q \to Q$ be a mapping defined by $Tq = (T_1 q, T_2 q, T q_3)$, where $T_1 q = P_{\tau_A F}(q_1 - \tau_A(Aq_1 + \Lambda^* q_3 + r\Lambda^*(\Lambda q_1 - q_2)))$, $T_2 q = P_{\tau_B G}(q_2 - \tau_B(Bq_2 - q_3 + r(q_2 - \Lambda T_1 q)))$ and $T_3 q = q_3 + r(\Lambda T_1 q - T_2 q)$ for $q = (q_1, q_2, q_3) \in Q$. Then T is nonexpansive, i.e.,

$$\|Tq - Tp\|_Q \leq \|q - p\|_Q$$

for all $q, p \in Q$.

Theorem 3 is the result of Wittmann [6]. This theorem is related to results of [1].

Theorem 3. *Let H be a Hilbert space and C be a closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that the fixed point of T is nonempty. Let $\{\alpha^{(k)}\}$ be a sequence of $[0, 1]$ such that*

$$\lim_{k \to 0} \alpha^{(k)} = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha^{(k)} = \infty$$

and

$$\sum_{k=0}^{\infty} |\alpha^{(k+1)} - \alpha^{(k)}| < \infty.$$

Let $\{x^{(k)}\}$ be an iterative sequence of C defined as follows: $x^{(0)} = x \in C$ and

$$x^{(k+1)} = \alpha^{(k)} x + (1 - \alpha^{(k)}) T x^{(k)}$$

for $k = 0, 1, 2, \ldots$. Then $\{x^{(k)}\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}x$ is the metric projection of H onto $F(T)$.

3 Main reslut

Theorem 4. *Let V and H be Hilbert spaces. Let $A : V \to V$ be a σ_A-inverse-storongly monotone mapping and $B : H \to H$ be a σ_B-inverse-strongly monotone mapping. Let $F : V \to (-\infty, \infty]$ and $G : H \to (-\infty, \infty]$ be proper convex lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of Λ, i.e., $\langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H$ for all $x \in H$ and $v \in V$. In addition, we assume that the operator $\Lambda^*\Lambda$ is a canonical isomorphism, i.e., $v = \Lambda^*\Lambda v$ for all $v \in V$. Let $Q = V \times H \times H$ be the Hilbert space with inner product*

$$\langle \cdot, \cdot \rangle_Q = \frac{1 - \tau_A r}{\tau_A} \langle \cdot, \cdot \rangle_V + \frac{1}{\tau_B} \langle \cdot, \cdot \rangle_H + \frac{1}{r} \langle \cdot, \cdot \rangle_H,$$

where τ_A, τ_B and r are positive constants satisfying $\tau_A r < 1$,

$$\tau_A < \frac{2\sigma_A}{2\sigma_A r + 1} \quad \text{and} \quad \tau_B < \frac{2\sigma_B}{2\sigma_B r + 1}.$$

Let $T : Q \to Q$ be a mapping defined by $Tq = (T_1 q, T_2 q, T_3 q)$, where

$$T_1 q = P_{\tau_A F}(q_1 - \tau_A(Aq_1 + \Lambda^* q_3 + r\Lambda^*(\Lambda q_1 - q_2))),$$

$$T_2 q = P_{\tau_B G}(q_2 - \tau_B(Bq_2 - q_3 + r(q_2 - \Lambda T_1 q))),$$

$$T_3 q = q_3 + r(\Lambda T_1 q - T_2 q)$$

for $q = (q_1, q_2, q_3) \in Q$. Assume that $F(T)$ is nonempty. Let $\{q^{(k)}\}$ be the sequence constructed by $q^{(0)} = q_0 \in Q$ and

$$q^{(k+1)} = \alpha^{(k)} q_0 + (1 - \alpha^{(k)}) T q^{(k)}$$

for $k = 0, 1, 2, \ldots$, where $\{\alpha^{(k)}\}$ be a sequence in $[0, 1]$ such that

$$\lim_{k \to 0} \alpha^{(k)} = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha^{(k)} = \infty$$

and

$$\sum_{k=0}^{\infty} |\alpha^{(k+1)} - \alpha^{(k)}| < \infty.$$

Then this iterative sequence $\{q^{(k)}\}$ converges strongly to q^* in Q as $k \to \infty$, q^* is a fixed point of T. Moreover the first component u in $q^* = (u, y, \lambda)$ is a solution of the problem (1).

Proof. By Theorem 2, T is nonexpansive. By Theorem 3, we obtain that the sequence $\{q^{(k)}\}$ converges strongly to a fixed point $q^* = (u, y, \lambda)$ of T. By Theorem 1, we have u is a solution of the problem $\langle Au, v - u \rangle_V + \langle \Lambda^* B \Lambda u, v - u \rangle_V + G(\Lambda v) - G(\Lambda u) + F(v) - F(u) \geq 0$ for all v in V. This completes the proof. □

4 Appendix

In this section, to sake of completeness, we show the proof of Theorems 1 and 2.

Proof of Theorem 1. Let $q = (u, y, \lambda)$ be a fixed point of T. Then

$$u = T_1 q, \quad y = T_2 q, \quad \lambda = T_3 q.$$

By the definition of T_3, we have

$$\lambda = \lambda + r(\Lambda u - y).$$

Then $\Lambda u - y = 0$, and

$$y = \Lambda u.$$

By the definition of T_1, we have

$$u = P_{\tau_A F}(u - \tau_A(Au + \Lambda^* \lambda + r\Lambda^*(\Lambda u - y))).$$

Then we have

$$\begin{aligned} &\langle u - (u - \tau_A(Au + \Lambda^*\lambda + r\Lambda^*(\Lambda u - y))), \\ &v - u \rangle_V + \tau_A F(v) - \tau_A F(u) \geq 0 \end{aligned}$$

for all $v \in V$. Since $\Lambda u - y = 0$, we have

$$\langle Au + \Lambda^*\lambda, v - u \rangle_V + F(v) - F(u) \geq 0$$

for all $v \in V$. Hence we have

$$F(v) \geq \langle -Au - \Lambda^*\lambda, v - u \rangle_V + F(u)$$

for all $v \in V$. This implies

$$-Au - \Lambda^*\lambda \in \partial F(u).$$

By the definition of T_2, we have

$$y = P_{\tau_B G}(y - \tau_B(By - \lambda + r(y - \Lambda u))).$$

Then we have

$$\langle y - (y - \tau_B(By - \lambda + r(y - \Lambda u))), z - y\rangle_H + \tau_B G(z) - \tau_B G(y) \geq 0$$

for all $z \in H$. Since $y - \Lambda u = 0$, we have

$$\langle By - \lambda, z - y\rangle_H + G(z) - G(y) \geq 0$$

for all $z \in H$. Hence we have

$$G(z) \geq \langle \lambda - By, z - y\rangle_H + G(y)$$

for all $z \in H$. This implies

$$\lambda - By \in \partial G(y).$$

We also obtain that if $-Au - \Lambda^*\lambda \in \partial F(u)$, $\lambda - By \in \partial G(y)$ and $y = \Lambda u$, then $q = (u, y, \lambda)$ is a fixed point of T. Let $q = (u, y, v)$ be a fixed point of T. Then we obtain that for all $v \in V$,

$$\langle Au + \Lambda^*\lambda, v - u\rangle_V + F(v) - F(u) \geq 0 \quad (3)$$

and

$$\langle By - \lambda, \Lambda v - y\rangle_H + G(\Lambda v) - G(y) \geq 0.$$

Since $y = \Lambda u$, we have

$$\begin{aligned}&\langle By - \lambda, \Lambda v - y\rangle_H + G(\Lambda v) - G(y)\\ &= \langle B\Lambda u - \lambda, \Lambda v - \Lambda u\rangle_H + G(\Lambda v) - G(\Lambda u)\\ &= \langle \Lambda^* B\Lambda u - \Lambda^*\lambda, v - u\rangle_V + G(\Lambda v) - G(\Lambda u).\end{aligned}$$

Then we have

$$\langle \Lambda^* B\Lambda u - \Lambda^*\lambda, v - u\rangle_V + G(\Lambda v) - G(\Lambda u) \geq 0 \quad (4)$$

for all $v \in V$. Adding (3) and (4), we find that u satisfies the inequality (1). □

To prove Theorem 2, we need the following: For all $\epsilon > 0$ and $a, b \in V$, we have

$$\langle a, b\rangle_V = \frac{1}{2\epsilon}\|a\|_V^2 + \frac{\epsilon}{2}\|b\|_V^2 - \frac{1}{2\epsilon}\|a - b\|_V^2. \quad (5)$$

Proof of Theorem 2. Define a mapping S_A of V into V by $S_A v = (1 - \tau_A r)v$ for $v \in V$. Then we obtain that for all $q_1, p_1 \in V$,

$$\begin{aligned}&\|S_A q_1 - S_A p_1\|_V^2\\ &= \|(1 - \tau_A r)(q_1 - p_1) - \tau_A(Aq_1 - Ap_1)\|_V^2\\ &= (1 - \tau_A r)^2\|q_1 - p_1\|_V^2\\ &\quad - 2\tau_A(1 - \tau_A r)\langle Aq_1 - Ap_1, q_1 - p_1\rangle_V\\ &\quad + \tau_A^2\|Aq_1 - Ap_1\|_V^2\\ &\leq (1 - \tau_A r)^2\|q_1 - p_1\|_V^2\\ &\quad - \delta_A\langle Aq_1 - Ap_1, q_1 - p_1\rangle_V, \qquad (6)\end{aligned}$$

where $\delta_A = \tau_A(1 - \tau_A r)\left(2 - \frac{\tau_A}{\sigma_A(1 - \tau_A r)}\right)$. By (2) and (5) with $\epsilon = 1 - \tau_A r$, $a = S_A q_1 - S_A p_1$ and $b = T_1 q - T_1 p$, we have

$$\begin{aligned}&\|T_1 q - T_1 p\|_V^2\\ &= \|P_{\tau_A F}(S_A q_1 - \tau_A\Lambda^*(q_3 - rq_2))\\ &\quad - P_{\tau_A F}(S_A p_1 - \tau_A\Lambda^*(p_3 - rp_2))\|_V^2\\ &\leq \langle T_1 q - T_1 p, S_A q_1 - S_A p_1\rangle_V\\ &\quad - \tau_A\langle T_1 q - T_1 p, \Lambda^*(q_3 - p_3) - r\Lambda^*(q_2 - p_2)\rangle_V\\ &= \frac{1}{2(1 - \tau_A r)}\|S_A q_1 - S_A p_1\|_V^2\\ &\quad + \frac{1 - \tau_A r}{2}\|T_1 q - T_1 p\|_V^2\\ &\quad - \frac{1 - \tau_A r}{2}\|(S_A q_1 - S_A p_1) - \epsilon(T_1 q - T_1 p)\|_V^2\\ &\quad - \tau_A\langle T_1 q - T_1 p, \Lambda^*(q_3 - p_3) - r\Lambda^*(q_2 - p_2)\rangle_V\end{aligned}$$

for all $q = (q_1, q_2, q_3)$, $p = (p_1, p_2, p_3) \in Q$. Then, by (8) and (5) with $\epsilon = 1$, we have

$$\begin{aligned}&\frac{1}{(1 - \tau_A r)\tau_A}\|(S_A q_1 - S_A p_1) - \epsilon(T_1 q - T_1 p)\|_V^2\\ &\quad + \frac{1 + \tau_A r}{2\tau_A}\|T_1 q - T_1 p\|_V^2\\ &\leq \frac{1}{2(1 - \tau_A r)\tau_A}\|S_A q_1 - S_A p_1\|_V^2\\ &\quad - \langle T_1 q - T_1 p, \Lambda^*(q_3 - p_3) - r\Lambda^*(q_2 - p_2)\rangle_V\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1-\tau_A r}{2\tau_A}\|q_1-p_1\|_V^2 \\
&\quad - \frac{\delta_A}{2}\langle Aq_1-Ap_1, q_1-p_1\rangle_V \\
&\quad - \langle T_1q-T_1p, \Lambda^*(q_3-p_3) - r\Lambda^*(q_2-p_2)\rangle_V \\
&= \frac{1-\tau_A r}{2\tau_A}\|q_1-p_1\|_V^2 \\
&\quad - \frac{\delta_A}{2}\langle Aq_1-Ap_1, q_1-p_1\rangle_V \\
&\quad - \langle T_1q-T_1p, \Lambda^*(q_3-p_3)\rangle_V \\
&\quad + r\langle \Lambda(T_1q-T_1p), q_2-p_2\rangle_H \\
&= \frac{1-\tau_A r}{2\tau_A}\|q_1-p_1\|_V^2 \\
&\quad - \frac{\delta_A}{2}\langle Aq_1-Ap_1, q_1-p_1\rangle_V \\
&\quad - \langle T_1q-T_1p, \Lambda^*(q_3-p_3)\rangle_V \\
&\quad + \frac{r}{2}\|\Lambda(T_1q-T_1p)\|_H^2 + \frac{r}{2}\|q_2-p_2\|_H^2 \\
&\quad - \frac{r}{2}\|\Lambda(T_1q-T_1p)-(q_2-p_2)\|_H^2
\end{aligned}
$$

for all $q, p \in Q$. Therefore we obtain that

$$
\begin{aligned}
&\frac{1}{2(1-\tau_A r)\tau_A}\|(S_Aq_1-S_Ap_1) \\
&\quad - (1-\tau_A r)(T_1q-T_1p)\|_V^2 \\
&\quad + \frac{1+\tau_A r}{2\tau_A}\|T_1q-T_1p\|_V^2 \\
&\quad + \frac{r}{2}\|\Lambda(T_1q-T_1p)-(q_2-p_2)\|_H^2 \\
&\quad + \frac{\delta_A}{2}\langle Aq_1-Ap_1, q_1-p_1\rangle_V \\
&\le \frac{1-\tau_A r}{2\tau_A}\|q_1-p_1\|_V^2 \\
&\quad - \langle \Lambda(T_1q-T_1p), q_3-p_3\rangle_H \\
&\quad + \frac{r}{2}\|\Lambda(T_1q-T_1p)\|_H^2 + \frac{r}{2}\|q_2-p_2\|_H^2 \qquad (7)
\end{aligned}
$$

for all $q, p \in Q$. Define a mapping S_B of H into H by $S_Bx = (1-\tau_B r)x - \tau_B Bx$ for $x \in H$. Then we obtain that for all $q_2, p_2 \in H$,

$$
\begin{aligned}
&\|S_Bq_2-S_Bp_2\|_H^2 \\
&= \|(1-\tau_B r)^2\|q_2-p_2\|_H^2 \\
&\quad - 2\tau_B(1-\tau_B r)\langle Bq_2-Bp_2, q_2-p_2\rangle_H \\
&\quad + \tau_B^2\|Bq_2-Bp_2\|_H^2 \\
&\le (1-\tau_B r)^2\|q_2-p_2\|_H^2 \\
&\quad - \tau_B(1-\tau_B r)\delta_B\langle Bq_2-Bp_2, q_2-p_2\rangle_H, \qquad (8)
\end{aligned}
$$

where $\delta_B = 2 - \frac{\tau_B}{\sigma_B(1-\tau_B r)}$. By (2) and (5) with $\epsilon = 1-\tau_B r$, $a = S_Bq_2 - S_Bp_2$ and $b = T_2q - T_2p$, we have

$$
\begin{aligned}
&\|T_2q-T_2p\|_H^2 \\
&= \|P_{\tau_B G}(S_Bq_2 + \tau_B r_1 q + \tau_B q_3) \\
&\quad - P_{\tau_B G}(S_Bp_2 + \tau_B r_1 p + \tau_B p_3)\|_H^2 \\
&= \langle T_2q-T_2p, S_Bq_2-S_Bp_2\rangle_H \\
&\quad + \tau_B\langle T_2q-T_2p, r\Lambda(T_1q-T_1p) + (q_3-p_3)\rangle_H \\
&\le \frac{1}{2(1-\tau_B r)}\|S_Bq_2-S_Bp_2\|_H^2 \\
&\quad + \frac{1-\tau_B r}{2}\|T_2q-T_2p\|_H^2 \\
&\quad - \frac{1}{2(1-\tau_B r)}\|(S_Bq_2-S_Bp_2) \\
&\quad - (1-\tau_B r)(T_2q-T_2p)\|_H^2 \\
&\quad + \tau_B\langle T_2q-T_2p, r\Lambda(T_1q-T_1p) + (q_3-p_3)\rangle_H
\end{aligned}
$$

Then, by (8) and (5) with $\epsilon = 1$, we have

$$
\begin{aligned}
&\frac{1}{2(1-\tau_B r)\tau_B}\|(S_Bq_2-S_Bp_2) \\
&\quad - (1-\tau_B r)(T_2q-T_2p)\|_H^2 \\
&\quad + \frac{1+\tau_B r}{2\tau_B}\|T_2q-T_2\|_H^2 \\
&\le \frac{1}{2(1-\tau_B r)\tau_B}\|S_Bq_2-S_Bp_2\|_H^2 \\
&\quad + r\langle T_2q-T_2p, \Lambda(T_1q-T_1p)\rangle_H \\
&\quad + \langle T_2q-T_2p, q_3-p_3\rangle_H \\
&\le \frac{1-\tau_B r}{2\tau_B}\|q_2-p_2\|_H^2 \\
&\quad - \frac{\delta_B}{2}\langle Bq_2-Bp_2, q_2-p_2\rangle_H \\
&\quad + r\langle T_2q-T_2p, \Lambda(T_1q-T_1p)\rangle_H \\
&\quad + \langle T_2q-T_2p, q_3-p_3\rangle_H \\
&= \frac{1-\tau_B r}{2\tau_B}\|q_2-p_2\|_H^2 \\
&\quad - \frac{\delta_B}{2}\langle Bq_2-Bp_2, q_2-p_2\rangle_H \\
&\quad + \frac{r}{2}\|T_2q-T_2p\|^2 + \frac{r}{2}\|\Lambda(T_1q-T_1p)\|_H^2 \\
&\quad - \frac{r}{2}\|(T_2q-T_2p)-\Lambda(T_1q-T_1p)\|_H^2 \\
&\quad + \langle T_2q-T_2p, q_3-p_3\rangle_H
\end{aligned}
$$

for all $q, p \in Q$. Therefore we obtain that

$$\begin{aligned}
&\frac{1}{2(1-\tau_B r)\tau_B}\|(1-\tau_B r)\left((q_2-p_2)\right.\\
&\quad \left.-(T_2q-T_2p)\right)-\tau_B(Bq_2-Bp_2)\|_H^2\\
&\quad +\frac{1+\tau_B r}{2\tau_B}\|T_2q-T_2p\|_H^2\\
&\quad +\frac{r}{2}\|(T_2q-T_2p)-\Lambda(T_1q-T_1p)\|_H^2\\
&\quad +\frac{\delta_B}{2}\langle Bq_2-Bp_2, q_2-p_2\rangle_H\\
&\le \frac{1-\tau_B r}{2\tau_B}\|q_2-p_2\|_H^2+\langle T_2q-T_2p, q_3-p_3\rangle_H\\
&\quad +\frac{r}{2}\|T_2q-T_2p\|_H^2+\frac{r}{2}\|\Lambda(T_1q-T_1p)\|_H^2
\end{aligned} \tag{9}$$

for all $q, p \in V$. For all $q, p \in V$, we have

$$\begin{aligned}
&\|T_3q-T_3p\|_H^2\\
&=\|q_3-p_3\|_H^2\\
&\quad +2r\langle q_3-p_3, \Lambda(T_1q-T_1p)-(T_2q-T_2)p\rangle_H\\
&\quad +r^2\|\Lambda(T_1q-T_1p)-(T_2q-T_2p)\|_H^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{1}{2r}\|T_3q-T_3p\|_H^2\\
&=\frac{1}{2r}\|q_3-p_3\|_H^2\\
&\quad +\langle q_3-p_3, \Lambda(T_1q-T_1p)-(T_2q-T_2p)\rangle_H\\
&\quad +\frac{r}{2}\|\Lambda(T_1q-T_1p)-(T_2q-T_2p)\|_H^2
\end{aligned} \tag{10}$$

for all $q, p \in Q$. By (7), (9) and (10), we have

$$\begin{aligned}
&\|Tq-Tp\|_Q^2+\delta_A\langle Aq_1-Ap_1, q_1-p_1\rangle_V\\
&\quad +\delta_B\langle Bq_2-Bp_2, q_2-p_2\rangle_H\\
&\quad +\frac{1}{(1-\tau_A r)\tau_A}\times\\
&\quad \|(1-\tau_A r)(q_1-p_1)-\tau_A(Aq_1-Ap_1)\|_V^2\\
&\quad +\frac{1}{(1-\tau_B r)\tau_B}\times\\
&\quad \|(1-\tau_B r)((q_2-p_2)-(T_2q-T_2p))\|_H^2\\
&\le \|q-p\|_Q^2
\end{aligned}$$

for all $q, p \in Q$. Therefore T is nonexpansive. □

5 Further topic

In [2], Badriev and Zadvornov consider a variational inequality problem for only one monotone mapping A. The authors show that for a strongly monotone Lipschitz continuous mapping A, an iterative sequence converges strongly to a solution. But using Theorem 3, we may obtain the strong convergence for an inverse-strongly monotone mapping A. This is a further topic.

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